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MARKET LIQUIDITY AND MARKET MAKING – THE CASE OF FIXED INCOME AND LOW INTERST RATES

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RESUME

Markedslikviditet og market making på obligationsmarkedet i et lavrentemiljø
Markedslikviditet har været meget i fokus på det seneste, især på obligationsmarkedet. Niveauet af markedslikviditet analyseres i en lagermodel for market making, der udvides til at kunne beskrive obligationsmarkedet. I modellen stiller market makerne likviditet til rådighed for markedet, hvilket de kompenseres for via forskellen mellem deres købs- og salgspriser, bid ask-spændet. Dette spænd er et ofte anvendt mål for markedslikviditet. Det vises, at lavrentemiljøer under visse betingelser kan være kendetegnet ved større bid ask-spænd – og dermed lavere markedslikviditet – sammenlignet med perioder med et højere renteniveau. Kapitaltab på market makernes lagerbeholdning af obligationer er mere sandsynlige i et lavrentemiljø, end når renteniveauet er højere, hvilket giver et større bid ask-spænd.

ABSTRACT

Market liquidity and market making – the case of fixed income and low interest rates
Market liquidity has received a lot of attention lately, especially in fixed-income markets. This paper studies the determinants of market liquidity in a theoretical model for market making with inventory costs, which is extended to the case of fixed-income securities. In the model, market makers provide liquidity to the market, which they are compensated for through the difference between the prices at which they buy and sell, the bid-ask spread. This spread is an often used measure for market liquidity. It is shown that under certain conditions, environments with low short-term interest rates can be characterised by lower market liquidity through wider bid-ask spreads compared to environments with higher interest rates. When interest rates are low, capital losses on the market makers' inventory holdings are more likely than when interest rates are higher, which leads to wider bid-ask spreads.

KEY WORDS

Market liquidity; market making; inventory costs; bid-ask spread; fixed income; low interest rates.

JEL CLASSIFICATION

G12; G20; E43.

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Market Liquidity and Market Making – the Case of Fixed Income and Low Interest Rates

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December 2015

Abstract

Market liquidity has received a lot of attention lately, especially in fixed-income markets. This paper studies the determinants of market liquidity in a theoretical model for market making with inventory costs, which is extended to the case of fixed-income securities. In the model, market makers provide liquidity to the market, which they are compensated for through the difference between the prices at which they buy and sell, the bid-ask spread. This spread is an often used measure for market liquidity. It is shown that under certain conditions, environments with low short-term interest rates can be characterised by lower market liquidity through wider bid-ask spreads compared to environments with higher interest rates. When interest rates are low, capital losses on the market makers' inventory holdings are more likely than when interest rates are higher, which leads to wider bid-ask spreads.

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1. Introduction

Market liquidity is important for the functioning of financial markets. It can be defined as the ability to buy or sell any amount of a security quickly, anonymously, with low price impact and at low costs. If market liquidity is low then trading is costly or, in extreme cases, impossible. Market liquidity in fixed-income markets is of particular interest for central banks, as monetary policy affects the real economy to a large extent through changes in interest rates. Also, fixed-income assets often serve as collateral for monetary-policy loans. In this paper, we study the determinants of market liquidity in a theoretical framework for market making. We show that under certain conditions, environments with low short-term interest rates can be characterised by lower market liquidity compared to environments with higher interest rates. And near a lower bound for the short interest rate market liquidity can be even lower due to the asymmetric distribution of future bond prices.

Liquidity in financial markets, especially fixed-income markets, has received renewed attention lately, cf. IMF (2015), Committee on the Global Financial System (2014) and Danmarks Nationalbank (2015, Chapter 3) for recent empirical studies. Events such as the October 2014 Treasury bond flash rally in the United States, in which the 10-year benchmark Treasury security experienced a 37 basis point trading range, only to close 6 basis points below its opening level without any significant news announcements, or the so-called taper tantrum during 2013 in the United States in which the announcement of near-future normalisation of unconventional monetary policy led to increasing market volatility and a worsening of market liquidity, are signs that market liquidity can disappear even for the most liquid assets.

Market makers are financial institutions which stand ready to buy and sell financial securities at given prices. They post different prices for buyers and sellers, respectively. They sell to the ask price and buy to the bid price, and they derive revenues from the spread between these two prices, the bid-ask spread. The bid-ask spread is a typical metric for market liquidity in the academic literature, see Roll (1984). A large spread reflects that it is expensive for potential buyers and sellers to act in the market. Illiquid markets are thus characterised by having large bid-ask spreads.

In this paper, we consider the determination of the bid-ask spread in a theoretical framework with a particular focus on the market for fixed-income securities. The research questions we have in mind are: Which factors are likely to affect the bid-ask spread? How is the current low interest rate environment likely to affect the spread? What happens to the spread when the short rate is near a lower bound?

Traditionally, the literature has pointed to three factors, which are likely to determine the size of the bid-ask spread: Order processing costs, adverse selection, and inventory costs. We will in what follows study the determination of the bid-ask spread within a theoretical framework where it is costly for market makers to hold securities in their inventories. Our starting point is a simple model for the determination of the bid-ask spread from Shen and Starr (2002). We extend the model in order to analyse the case of fixed-income securities. This extension of the model enables us to provide deeper insight into factors which are likely to determine the bid-ask spread. Specifically, we can analyse inventory risk and its likely impact on the bid-ask spread, and thus on market liquidity.

We find that the bid-ask spread is adjusted to cover two terms: the expected costs at expected trading volume and the expected (net) holding period return from holding the inventory for one period and selling it again the following period. The second term implies that expectations of a higher future bond price depress the bid-ask spread all else being equal; higher expected compensation for holding the inventory puts downward pressure on the bid-ask spreads, the intuition being that in expectation terms the market maker will gain on his market making, which through free entry and perfect competition forces the spread down such that the market maker does not lose clients. We then show that under the assumption of no-arbitrage and in the case in which the market makers demand exactly the same risk-adjusted return as the bond market in general, the second term does not in expectation terms affect the bid-ask spread. The intuition is that market makers are already fully compensated for the inherent risk of changes in bond prices in risk-adjusted expectation terms through the market price of the bond, and therefore they do not pass expected capital gains or losses on to the bid-ask spread in equilibrium.

We next assume that market makers demand less compensation for bearing risk and we introduce an equilibrium bond-pricing model along the lines of Cox et al. (1985); i.e. the so-called CIR-model. We are to our knowledge the first to do so. In this case we show that for given costs, low interest rate environments are characterised by higher bid-ask spreads than environments with higher interest rates. This is because market makers extract a positive compensation for holding risky securities, which in equilibrium results in lower bid-ask spreads, and this compensation is increasing in the level of the interest rate.

We lastly analyse market liquidity at or near the lower bound. We show that here, the distribution of future bond prices is asymmetric in the sense that bond prices are most likely to fall in contrast to the case outside the lower bound where bond-price gains and losses are equally likely. And when market makers are loss averse in the sense that capital losses from holding an inventory of bonds are more costly than gains of equivalent size, bid-ask spreads are largest at the lower bound. This holds even when market makers demand exactly the same risk-adjusted return as the market in general.

The remainder of this paper is structured as follows: Section (2) discusses different concepts of liquidity and the importance of market making for market liquidity. Section (3) sets up a model for a market maker and the determination of the bid-ask spread. We look at the case of the fixed-income market in section (4), and analyse the impact on the bid-ask spread of low levels of the short rate, and the effect of asymmetry distribution for bond prices at or near the lower bound. Section (5) concludes.

2. Market liquidity, market making and bid-ask spreads

2.1. What is liquidity?

Liquidity relates to at least three different concepts: monetary liquidity, market liquidity and funding liquidity. Whereas *monetary liquidity* relates to monetary-policy aggregates, this paper primarily deals with *market liquidity*, which can be defined as the ability to buy and sell arbitrary amounts of a security quickly, anonymously, with low impact on market prices and at low costs, cf. Campbell et al. (1997). *Funding liquidity*, on the other hand, relates to the ability of market participants to obtain funding at acceptable conditions and should not be confused with market liquidity, cf. IMF (2015). Funding liquidity is for example important for banks, which receive deposits that quickly can be withdrawn, and lend on a longer term basis, see Foucault et al. (2013).

The liquidity concepts are interrelated through various channels. For example, funding liquidity is important for market liquidity, cf. Dick-Nielsen et al. (2014). In the model presented below, funding liquidity is a prerequisite for market liquidity, because market makers need access to funding in order to build up and maintain an inventory of securities. Easy access to funding then implies higher market liquidity. However, the relationship between funding liquidity and market liquidity can also be reversed, i.e. market liquidity can be a prerequisite for funding liquidity, e.g. if pledging of collateral is required for trading with financial assets. The required amount of collateral typically depends on how liquid the asset is, and therefore higher market liquidity can result in cheaper funding and higher funding liquidity, cf. also Brunnermeier and Pedersen (2009).

The definition of market liquidity has many dimensions, and therefore it is hard to capture by any single measure. Various empirical measures have been suggested in the literature like the median trade size, the turnover rate, bid-ask spreads, and the trade price impact measure, see also Buchholst et al. (2010) or IMF (2015) for an extensive list. Among the most applied measures are the effective bid-ask spreads (Roll, 1984) and price impact measures (Amihud, 2002), which are the most widely used metrics for market liquidity in the academic literature. In this paper, we will follow the academic literature and use the bid-ask spread as the starting point for our analysis.

2.2. The role of market makers

In what follows we will look at the role played by market makers in financial markets with emphasis on fixed-income markets. From a central bank perspective, fixed income markets are of special interest, as the monetary transmission mechanism works through interest rates. Furthermore, fixed-income assets often serve as collateral for monetary-policy loans.

One feature of bond markets that limits their liquidity is that individual issuers may have a large number of different securities outstanding. This makes bonds a heterogeneous asset class in which many securities can be thinly traded. At the same time, institutional investors often hold assets to maturity, and when they do trade, they do so in large amounts. Thus, trading in any individual issue is often infrequent and lumpy. This reduces the probability of matching buyers and sellers of any given bond at any given time. For that reason, bond markets, particularly those for corporate issues, tend to rely on market makers.

A market maker serves as an intermediary between buyers and sellers of securities. Market makers are therefore important for establishing equilibrium in financial markets. Market makers play an important role for market liquidity. How liquid a market for a specific security is, depends on whether market makers can and will counteract imbalances between the number of potential buyers and sellers. They might do so by buying and selling the security themselves and thereby absorbing a given order flow using their inventory as a buffer. By standing ready to buy and sell the security, the intermediary makes a market. An inventory of securities and large risk-bearing capacity are necessary for market makers in order to meet supply and demand in a situation with more buyers than sellers or vice versa, in which case the market makers must step in to absorb the excess flow, cf. Fender and Lewrick (2015) and Committee on the Global Financial System (2014).

Market makers offer different prices to potential buyers and sellers, i.e. they sell at the announced *ask price* and they buy at the announced *bid price*. The spread between these two prices, the *bid-ask spread*, is the market maker's compensation for facilitating market liquidity. The inverse of the bid-ask spread is an often used measure of market liquidity, cf. Roll (1984). A large spread implies that it is expensive for buyers and sellers to act in the market. Illiquid markets are characterised by large bid-ask spreads, cf. Shen and Starr (2002).

In standard models of market making, the costs of acting as a market maker can be divided into three subgroups: order-processing costs, adverse-selection costs, and inventory costs, cf. Campbell et al. (1997). Order-processing costs cover basic setup and operating costs of trading, such as manpower, computers etc. Adverse-selection costs arise when some investors are better informed about the security's true value than the market maker, and therefore trading with these investors will, on average, result in losses for which the market makers must be compensated. In this paper, we primarily focus on the third type of costs, inventory costs, which involves the costs and risks facing market makers who must maintain a sufficient inventory of securities on their balance sheets in order to facilitate market liquidity. Since the market maker requires compensation for bearing these costs, they become important determinants of the bid-ask spread, and thus important determinants of the level of market liquidity. In models of market making, the costs of holding securities in inventory is often an important determinant for the bid-ask spread, cf. Stoll (1978) and Shen and Starr (2002).

A simple model of a market maker's profit is illustrated in figure (1). The decision for a market marker about the size of its bid-ask spread, and thereby market liquidity, and its

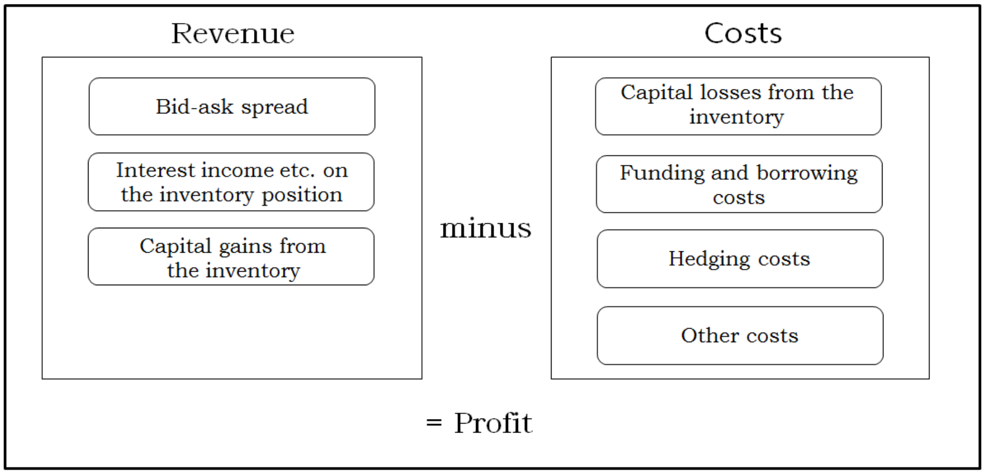

Figure 1: **Simplified model of a market maker's profit**

inventory, can be thought of as depending on the following list of factors:

- *Profit-loss on inventory*: The higher the price risk of the asset, the higher compensation the market maker must demand in form of bid-ask spreads. The market maker must be compensated for risk as it can be forced to hold assets in quantities that it does not want due to changing flows on its balance sheet. Even though the market maker is compensated for the price risk via the yield on the asset, the risk profile of the security affects the market maker's capital. The willingness to take on risk on the market maker's balance sheet depends on its preferences for risk, and this willingness can be crucial for the efficiency of the market, especially for less liquid markets. The possibility for taking on risk and hence having an inventory of assets also depends on the ability of loss absorption. For example, if the market marker is unwilling or unable to provide the necessary capital, its ability to absorb flows of assets is limited as the probability of not fulfilling capital requirements is larger.

- *Funding or opportunity costs*: A market maker must typically fund its positions. The higher the funding costs, the more expensive it is to hold a given inventory. Alternatively, the funding costs can be thought of as the opportunity costs of market making.

- *Hedging costs*: A market maker can wish to partly or fully hedge the risk inherent in its portfolio. This implies that the bid-ask spread depends on other financial markets such as an example the futures market and derivative markets.

One important concern is the possibility and ability to hedge the risk inherent in the inventory. This is discussed in the next section as this is key to the theoretical insights in this paper.

2.2.1 Hedging away the inventory risk?

The model we employ in this paper puts a special emphasis on inventory risks. If market makers were able and willing to perfectly hedge their positions they could eliminate all inventory risk. A market maker operating on the sovereign debt market has opportunities for hedging in e.g. the futures market. If the market maker receives a sovereign bond in its portfolio, the interest rate risk can be hedged by selling a future.

Even when it is possible to mitigate the inventory risks by hedging, it is often not possible to hedge the inventory risk perfectly, cf. Parameswaran (2003). As an example, there exits no possibilities for fully hedging inflation-linked bonds or corporate bonds, and mortgage bonds involve the risk of prepayments. Parameswaran (2003) lists four reasons why it can be impossible to hedge perfectly in practice using futures contracts:

1. The asset whose price risk the market participant wants to hedge against, may not be the same as the asset on which the futures contract is written.[1]
2. The exact date on which the asset will be bought or sold may not be known in advance.
3. Mismatch between the maturity of the asset and the futures contract.
4. The quantity which the market participant wants to hedge may not be equal to the quantity specified in the futures contract.

Due to these reasons, market participants who choose to hedge their position face so-called *basis risk*, which is the risk remaining on their position after the hedge.

Another question is whether market makers actually wish to fully hedge their positions as hedging is costly. Financial institutions can allocate capital and loss absorption capacity to their market making activities and choose to hold open positions and thereby increase risk and expected returns.

3. A model of market making – simple case

We next set up a theoretical model which in a consistent way can show how the bid-ask spread under given, stated assumptions is determined. The following model builds on Shen and Starr (2002) with a small extension since we explicitly take discounting into account.[2] At first we consider a simple case where the traded asset can be any type of security. We will within this framework introduce a quadratic cost function, as this gives extra insight into how bid-ask spreads are determined.

[1] In this case, the price risk can be hedged using a futures contract written on another asset. This is denoted *crosshedging*.

[2] It is usual in the market microstructure theory to measure the distance between two time periods in minutes. Hence, discounting is for simplicity left out of these models. Below we will incorporate into the market-making model a process for the short rate, which takes discounting into account, and therefore we also do so in this section for consistency.

We continue by analysing the case where the traded security is a bond, and finish by looking at the case where some lower bound for the short interest rate has been hit. In the case of a bond we proceed in two steps. During the first stage, we will analyse the determination of the bid-ask spread holding costs constant; that is we do not make any assumptions about the cost function. During the second stage, we will consider different analytical expressions for the cost function.

In the simple model, P_t denotes the price of a traded asset at time t. V_t^b and V_t^s denotes buy and sales volumes, respectively, end of period t from the perspective of the market maker. The market maker's net position of the security, N_t, evolves as follows:

$$N_{t+1} = N_t - V_t^s + V_t^b. \tag{1}$$

The market maker accepts all orders passively and adjusts the symmetrical spread (half of the bid-ask spread), S_t. The market maker has an initial cash position of M_t, carried over from previous periods, and the cost of providing liquidity, C_t, for the market maker. Π_t denotes the value of the market marker's position,

$$\Pi_t = P_t N_t + M_t, \tag{2}$$

while the cash position evolves as

$$M_{t+1} = (1 + S_t)\, P_t V_t^s - (1 - S_t)\, P_t V_t^b + (R_t^m)^{-1} M_t - C_{t+1}, \tag{3}$$

We will have more to say about these in the sections to come. R_t^m denotes the relevant discount factor for the market maker, which is also assumed to reflect the rate at which the market maker obtains funding.

We assume a market characterised by perfect competition and free entry. This implies a zero-profit condition in equilibrium, $R_t^m E_t[\Pi_{t+1}] - \Pi_t = 0$:

$$R_t^m E_t[\Pi_{t+1}] = R_t^m E_t[P_{t+1} N_{t+1} + M_{t+1}] = P_t N_t + M_t = \Pi_t. \tag{4}$$

Equation (4) implies that

$$R_t^m E_t[P_{t+1} N_{t+1} + M_{t+1}] - (P_t N_t + M_t) = (R_t^m E_t[P_{t+1}] - P_t)\, N_t + 2R_t^m S_t P_t V_0 - R_t^m E_t[C_{t+1}] = 0, \tag{5}$$

in which V_0 is the expected flow of the security in the inventory assumed equal for buy and sell orders. Expression (5) holds under the assumptions that the net position and the asset price are uncorrelated, and that the buy and sell flows follow known distributions. We next assume that the price of the traded asset follows a martingale in discounted terms, $R_t^m E_t[P_{t+1}] = P_t$. This implies that if the zero-profit condition, condition (4), needs to be fulfilled then the equilibrium bid-ask spread, $S_t^\star$, must be equal to

$$S_t^\star = \frac{E_t\left[C_{t+1}\right]}{2P_tV_0}. \tag{6}$$

The equilibrium spread is adjusted to cover expected costs at expected trading volume. This result is quite standard in economics: In markets characterised by free entry and perfect competition, prices (in this case the spread) are set equal to (expected) marginal costs.

3.1. Quadratic cost function

The last piece of the simple model is an assumption about costs, C_t. To get more insight into the determination of the bid-ask spread, we will follow Shen and Starr (2002) and assume the case of a quadratic cost function. Holding the security inventory requires financing, as pointed out in section (2.2), and the average cost of capital may vary with the size of the market maker's inventory position. When a market maker relies on self-financing, the cost of carrying inventory can be thought of as an opportunity cost. A quadratic cost function can also reflect that market makers have the capacity to hold risky assets in its inventory, but that this capacity is limited and costs increases faster than linearly in the value of the inventory.

We analyse the situation where costs are quadratic in the value of the inventory,

$$C_{t+1} \equiv \eta\left(P_{t+1}N_{t+1}\right)^2, \tag{7}$$

with $\eta > 0$. The economic intuition behind a quadratic cost function is that it is a simplified way of capturing risk averse market markers. Alternatively, the quadratic function can reflect an increasing risk premium if the market maker uses debt financing, which may be added to interest rates on lending to a market maker whose position is increasingly leveraged.

With a quadratic cost function, the equilibrium bid-ask spread can be written as

$$S_t^\star = \frac{\eta}{2P_tV_0}\left(P_t^2 + \sigma_p^2\right)\left(N_t^2 + 2\sigma_v^2 - 2\sigma_{V^S,V^B}\right), \tag{8}$$

using that $\eta E_t\left[\left(P_{t+1}N_{t+1}\right)^2\right] = \eta\left(P_t^2 + \sigma_p^2\right)E_t\left[\left(N_t - V_t^s + V_t^b\right)^2\right]$. We can thus conclude that in this model, the bid-ask spread is increasing in the risk inherited in the security, σ_p^2. This reflects that a risk averse market maker must be compensated for bearing price risk, and the riskier the asset, the greater is the compensation demanded in terms of a higher bid-ask spread. For the same reason, the equilibrium bid-ask spread is increasing in the volatility of the trading volume, σ_v^2, and decreasing in the covariance between buy and sell trading volumes, σ_{V^S,V^B}. Finally, the equilibrium bid-ask spread is increasing in the size of the inventory, N_t. This reflects that the costs are increasing in the size of the inventory risk: The larger the inventory, the higher are the costs and risk associated with holding the inventory, which the market maker needs to be compensated for. This is the simple case also studied in Shen and Starr (2002).

3.2. Fixed costs

In this subsection, we introduce a simple cost function as an alternative to the quadratic cost functions used in Shen and Starr (2002). As described in section (2), a market maker faces some costs that are not dependent on the value of its inventory position, i.e. fixed costs such as housing rents, wages, IT-systems etc. It is straightforward to extend the simple model to account for such fixed costs, that is

$$C_{t+1} \equiv F. \tag{9}$$

With this simple cost function, we can write the equilibrium bid-ask spread as

$$S_t^\star = \frac{F}{2P_t V_0}. \tag{10}$$

In this case, the bid-ask spread is unambiguously decreasing in the expected trading volume, V_0. This is intuitive since a market with a decreasing trading volume leaves the market maker with fewer transactions to cover his fixed costs, and so the compensation per trade, the bid-ask spread, must increase. Hence, if trading volumes drop, this version of the model predicts that market liquidity will be lower, due to a widening bid-ask spread. This interpretation seems in line with some of the observations in IMF (2015) and Committee on the Global Financial System (2014) that a falling trading volume can have an impact on market liquidity.

So far we have considered the simple case in which the traded asset could be any traded asset in the financial market. In the next section, we analyse the case of fixed-income securities more closely.

4. Market making – the case of fixed income

In this section, we look at the case in which the traded asset is a fixed-income security, e.g. a bond. Specifically, in the model P_t^n denotes the price of a zero-coupon, non-defaultable bond at time t with time-to-maturity n. Now, the market maker's net position of the security, N_t, evolves as

$$\Pi_t = P_t^n N_t + M_t. \tag{11}$$

In what follows, the initial net position is assumed to be positive, $N_t > 0$, reflecting that the market makers on average and in the aggregate are warehousing assets. The cash position evolves as

$$M_{t+1} = (1 + S_t)\, P_t^n V_t^s - (1 - S_t)\, P_t^n V_t^b + (R_t^m)^{-1} M_t - C_{t+1}. \tag{12}$$

In the case where the asset is a bond, the zero-profit condition is

$$R_t^m E_t [\Pi_{t+1}] = R_t^m E_t \left[P_{t+1}^{n-1} N_{t+1} + M_{t+1}\right] = P_t^n N_t + M_t = \Pi_t. \tag{13}$$

This implies that

$$R_t^m E_t \left[P_{t+1}^{n-1} N_{t+1} + M_{t+1}\right] - (P_t^n N_t + M_t) = \left(R_t^m E_t \left[P_{t+1}^{n-1}\right] - P_t^n\right) N_t + 2R_t^m S_t P_t^n V_0 - R_t^m E_t [C_{t+1}] = 0, \tag{14}$$

and hence that if the zero-profit condition needs to be fulfilled, then the equilibrium bid-ask spread must be given by

$$S_t^\star = \frac{E_t [C_{t+1}]}{2P_t^n V_0} - \frac{N_t}{2V_0} \frac{\left(R_t^m E_t \left[P_{t+1}^{n-1}\right] - P_t^n\right)}{R_t^m P_t^n}. \tag{15}$$

In general, expression (15) shows that the equilibrium bid-ask spread in the case of fixed-income securities is adjusted to cover two terms: the expected costs at expected trading volume, the first term in (15); and the expected, risk-adjusted (net) holding period return from holding the inventory for one period and selling it again in the following period, the second term in (15). This expression holds for any given cost function, C_t. To get more insight into the two terms determining the equilibrium bid-ask spread, we will proceed in two steps. At first we consider the determination of the bid-ask spread taking costs as given. During the second stage, we will consider different analytical expressions for the cost function.

Clearly, for given expected future costs, we get that

$$\left.\frac{\partial S_t^\star}{\partial E_t \left[P_{t+1}^{n-1}\right]}\right|_{costs} < 0.$$

The equilibrium bid-ask spread is thus decreasing in the expected future bond price when the bond price today and costs are taken as given and $N_t > 0$. Therefore, the spread will be higher in a case where capital losses are expected relative to a case where capital gains are expected. This result is intuitive since a market maker facing an expected capital gain on its inventory does not need to be compensated to the same extent as a market maker facing an expected capital loss, i.e. the bid-ask spread is lower when capital gains are expected. Hence, the market maker will lower the bid-ask spread if the market maker expects a gain on its portfolio and vice versa. The intuition is that in expectation terms the market maker will gain on his market making, like everybody else in the market, which through free entry and perfect competition will lead to downward pressure on spreads. In equilibrium, the market maker must lower his spread in order not to lose clients. Likewise, the market marker will increase the bid-ask spread to compensate for expected capital losses on its portfolio.

In the next subsection we will study the second term in expression (15) in greater detail by relaxing the assumption of exogenous bond pricing. Later we will consider both terms in (15) simultaneously, when we introduce different cost functions in the model with equilibrium bond prices.

4.1. The case where market makers demand exactly the same risk-adjusted return as the bond market

In the simple case in section (3), we assumed that the asset price followed a (discounted) martingale, which implies that the second term in (15) disappears. In the case of a bond, we have not so far made any assumptions about the future bond price and its expectation. To get more insight into the determination of the equilibrium bid-ask spread, we next assume that bonds are priced on a market without arbitrage opportunities.

At first, we consider the special case where market makers demand *exactly* the same risk-adjusted return on the bond holdings as the bond market in general. Below we will consider the case where market makers demand a lower risk-adjusted return than the bond market.

When bonds are priced on a market without arbitrage opportunities and when market makers demand exactly the same risk-adjusted return as the bond market in general, it can be shown that[3]

$$R_t^m E_t\left[P_{t+1}^{n-1}\right] = P_t^n \tag{16}$$

and thus, the second term in the determination of the bid-ask spread, expression (15), cancels out. This leaves us with the same result as Shen and Starr (2002), i.e. the equilibrium bid-ask spread is determined by

$$S_t^\star = \frac{E_t\left[C_{t+1}\right]}{2P_t V_0}. \tag{17}$$

Hence, in this special case, expected changes in the bond price do not affect the equilibrium bid-ask spread (for given costs). The intuition is that market makers are already fully compensated for the inherent risk of bond-price changes in risk-adjusted expectations terms through the market price of the bond, and therefore they do not pass expected capital gains/losses on to the bid-ask spread in equilibrium.

4.2. The case where market makers demand a lower risk-adjusted return than the bond market

We now turn to the case where market makers demand a lower risk-adjusted return than the bond market in general. This can, among other things, reflect that market makers on average

[3] Below we will assume a CIR-type process for the short rate. There we show that $E_t\left[P_{t+1}^{n-1}\right]/P_t^n = e^{(1-\lambda_0 B_{n-1}\sigma)r_t}$, see below for notation. When market makers demand exactly the same risk-adjusted return as the bond market in general, the relevant discount factor for the market maker is $R_t^m = e^{-r_t+\lambda_0 B_{n-1}\sigma r_t}$. Therefore, the second term in expression (15) vanishes since $\left(R_t^m E_t\left[P_{t+1}^{n-1}\right] - P_t^n\right) / \left(R_t^m P_t^n\right) = E_t\left[P_{t+1}^{n-1}\right]/P_t^n - \left(R_t^m\right)^{-1} = e^{(1-\lambda_0 B_{n-1}\sigma)r_t} - e^{(1-\lambda_0 B_{n-1}\sigma)r_t} = 0.$

have more risk-bearing capacity than e.g. non-financial firms and households that are active on the bond markets and/or that the market makers are less risk averse. Even in a bond market without arbitrage opportunities, we now get that $R_t^m E_t\left[P_{t+1}^{n-1}\right] \neq P_t^n$. Therefore, both terms in expression (15) now affect the determination of the equilibrium bid-ask spread.

Since market makers demand a lower risk-adjusted return than the bond market in general, i.e. they discount the future less heavily than the bond market, we get that $R_t^m E_t\left[P_{t+1}^{n-1}\right] > P_t^n$, see below for a proof within a CIR-type model. Hence, market makers are overcompensated for bearing the inventory risk in the sense that the expected risk-adjusted return exceeds what they require to hold the bond in inventory. When the initial inventory position is positive, $N_t > 0$, we see that this puts a downward pressure on the bid-ask spread, cf. expression (15). In the competitive equilibrium, the market maker must lower its bid-ask spread in order not to lose clients to competitors, i.e. the bid-ask spread is, all else equal, lower compared to the case where market makers demand exactly the same risk-adjusted return as the bond market.

In order to say more regarding the second term in (15), we need more structure on the model. More precisely we will extend the assumption of no arbitrage opportunities on the bond market and assume a specific process for the short rate and, thus, bond prices. The advantage of a model for bond prices is that it allows us to derive explicit expressions for prices and risk. We have left the details of the bond pricing model to appendix (A). Here we simply present the main insight.

The short rate is assumed to follow the stochastic process

$$r_{t+1} = \left(1 - \phi\right)\theta + \phi r_t + \sigma\sqrt{r_t}\varepsilon_{t+1}. \tag{18}$$

with $\varepsilon_{t+1} \sim N(0, 1)$. This is the discrete-time version of the CIR-model (Cox et al., 1985), cf. Sun (1992) and Backus et al. (1998). We have chosen this specification for several reasons: First, the model is well-known in the literature, and it provides a relatively simple expression for the risk-adjusted return. Second, it guarantees a positive interest rate, and consequently has a lower bound.[4] As shown in appendix, θ can be interpreted as the long-run mean of the short rate, ϕ determines the persistence of the short rate, and σr_t determines volatility. Notice that volatility depends upon the short rate such that higher levels of the short rate implies higher volatility, all else being equal. As shown below, this will become important for the results in this paper.

As shown in the appendix, the process for the short rate together with the assumption of no-arbitrage allows us to write the non-defaultable, zero-coupon bond price at time t with

[4] For the analysis in this paper it is not important whether the lower bound is zero or slightly negative.

time-to-maturity n, P_t^n, as follows

$$
\begin{aligned}
P_t^n &= e^{-A_n - B_n r_t} \\
A_{n+1} &= A_n + B_n\left(1-\phi\right)\theta, \qquad\qquad A_0 = 0, A_1 = 0 \\
B_{n+1} &= \left(1 + \lambda_0^2/2 + B_n\phi\right) - \left(\lambda_0 + B_n\sigma\right)^2/2, \; B_0 = 0, B_1 = 1
\end{aligned} \tag{19}
$$

Yields are given by $y_t^n = -\frac{1}{n}log(P_t^n)$. A_n determines the shape of the average yield curve. B_n determines by how much the yield curve changes if the short rate changes. The interpretation of the the parameter λ_0 is given below.

To keep things simple, in this subsection we focus on the second term in (15), i.e. costs are taken as given. Below we will relax this assumption. As shown in appendix (A), this interest-rate model allows us to derive an explicit expression for the term $\left(R_t^m E_t\left[P_{t+1}^{n-1}\right] - P_t^n\right)/R_t^m P_t^n$, i.e. the expected risk-adjusted capital gain, and therefore the second term in the determination of the equilibrium bid-ask spread, equation (15), can be written as

$$
-\frac{N_t}{2V_0}\frac{\left(R_t^m E_t\left[P_{t+1}^{n-1}\right] - P_t^n\right)}{R_t^m P_t^n} = -\frac{N_t}{2V_0}e^{r_t}\left(e^{-\lambda_0 B_{n-1}\sigma r_t} - 1\right), \tag{20}
$$

where we have imposed $R_t^m = e^{-r_t}$, i.e. the market maker's discount rate is the risk-free short rate. This reflects the assumption that market makers demand a lower risk-adjusted return than the bond market where the relevant discount rate is $r_t - \lambda_0 B_{n-1}\sigma r_t$, which is larger than r_t since $\lambda_0 < 0$, cf. below. It can be viewed as a normalisation that we assume market makers to be risk neutral. The qualitative results below will carry through as long as the market makers demand a lower risk-adjusted return than the bond market in general, i.e. the relevant discount rate for the market makers is less than $r_t - \lambda_0 B_{n-1}\sigma r_t$.

The term $-\lambda_0 B_{n-1}\sigma r_t$ can be viewed as an asset-specific time-varying risk premium. This term is positive since the process for the short rate, expression (18), guarantees a non-negative interest rate and given that λ_0 is negative. We assume $\lambda_0 < 0$ since this is a necessary assumption in the CIR-model in order to match an upward sloping yield curve on average. σr_t is the underlying risk or volatility in the economy, or the amount of risk. λ_0 is the price of risk which the market demands for holding one unit of risk. It consequently reflects how risk averse the bond market is; it does not say anything about the capacity for carrying risk. Finally, B_{n-1} determines by how much the specific asset, in this case a bond, depends of this value of risk.

Expression (20) is clearly decreasing in the initial interest rate, r_t, when the initial inventory position is positive, $N_t > 0$. Hence, ignoring the first term in equation (15) we find that low interest rate environments are characterised by higher bid-ask spreads, and thus lower market liquidity, than environments with higher interest rates. Two observations help explain the intuition behind this result: 1) the market maker extract a positive excess risk premium $-\lambda_0 B_{n-1}\sigma r_t > 0$, which in equilibrium results in a lower bid-ask spread, and 2) this risk premium is increasing in the level of the short rate. The latter observation follows directly from the CIR-model when $\lambda_0 < 0$, since the volatility of the short rate is increasing in the interest rate level, cf. expression (18). This property of the CIR-model seems to be supported empirically, see for example Piazzesi (2003). In sum, the risk premium is smaller when interest rates are low compared to when interest rates are high, and therefore market makers must to a higher extend be compensated for their costs through the bid-ask spread, resulting in a higher bid-ask spread.

4.2.1 Bid-ask spread and interest rate level – including the cost function

Above we analysed the effect of the interest rate level on the bid-ask spread, and thereby market liquidity, taking costs as given. In this subsection, we relax this assumption and investigate further both terms determining the equilibrium bid-ask spread in relation (15); i.e. we include the cost function in our analysis. To do so, we have to make further assumptions about its shape. We will consider two specifications of the cost function.

To be able to compare our results with the simple version of the model from Shen and Starr (2002), we first consider the same cost function as in section (3), i.e.

$$C_{t+1} \equiv \eta \left(P_{t+1}^{n-1} N_{t+1} \right)^2 , \tag{21}$$

where $\eta > 0$. We saw in section (3) that this cost function was a simple (and *ad hoc*) way to incorporate into the model some of the factors which are believed to affect market liquidity through the equilibrium bid-ask spread, e.g. bond-price risk and trading-volume risk. For our purpose, this specific cost function introduces some ambiguity when analysing the effect of low interest rates on market liquidity, as we will see below. Later, we will consider a related cost function, which does not have this property.

When market makers demand a lower risk-adjusted return than the bond market in general, and when costs are quadratic in the future value of the market maker's bond holdings, the equilibrium bid-ask spread from expression (15) can be rewritten as

$$S_t^\star = \frac{\eta}{2P_t^n V_0} \left(\left(E_t \left[P_{t+1}^{n-1} \right] \right)^2 + V_t \left[P_{t+1}^{n-1} \right] \right) \left(N_t^2 + 2\sigma_v^2 - 2\sigma_{V^S,V^B} \right) - \frac{N_t}{2V_0} e^{r_t} \left(e^{-\lambda_0 B_{n-1} \sigma r_t} - 1 \right). \tag{22}$$

Hence, we have an additional term compared to the expression determining the equilibrium bid-ask spread in the simple model in section (3), equation (8), reflecting the expected risk-adjusted capital gain on the inventory position. In appendix (B), we derive analytical expressions for the expected value and variance of the future bond price, $E_t \left[P_{t+1}^{n-1} \right]$ and $V_t \left[P_{t+1}^{n-1} \right]$, respectively.

With this cost function it is ambiguous whether a lower initial interest rate results in a larger bid-ask spread, and thus lower market liquidity. On the one hand, we know from above that the second term works in favour of this results, due to expected risk-adjusted capital gains being lower when interest rates are low. On the other hand, low interest rates will, *ceteris paribus*, lead to lower expected future bond prices, resulting in a smaller value of the future inventory. With the specified cost function, a smaller future value of the inventory implies lower costs, and therefore the bid-ask spread does not need to be as large in order to compensate the market maker for holding bonds in inventory. Also, the variance of future bond prices will be lower when interest rates are low, cf. above. In general, it is ambiguous which of these two counteracting effects are dominating, and therefore it is ambiguous whether lower initial interest rates result in higher bid-ask spread.

The ambiguity arises due to the specific formulation of the cost function. If we make a minor change in the timing assumption when specifying the cost function, low interest rates are unambiguously associated with higher bid-ask spread, and thus lower market liquidity, than high interest rates. To see this, assume that costs are quadratic in the value of the market maker's bond holdings this period, i.e.

$$C_{t+1} \equiv \eta \left(P_t^n N_t\right)^2 . \tag{23}$$

Now, the equilibrium bid-ask spread simplifies to

$$S_t^{\star} = \frac{\eta}{2V_0} P_t^n N_t^2 - \frac{N_t}{2V_0} e^{r_t} \left(e^{-\lambda_0 B_{n-1} \sigma r_t} - 1\right) = \frac{\eta}{2V_0} e^{-A_n - B_n r_t} N_t^2 - \frac{N_t}{2V_0} e^{r_t} \left(e^{-\lambda_0 B_{n-1} \sigma r_t} - 1\right) \tag{24}$$

where we use that $P_t^n = e^{-A_n - B_n r_t}$ in the CIR-model. This version of the equilibrium bid-ask spread is unambiguously decreasing in the level of the initial interest rate. That is, low interest rate environments are characterised by lower market liquidity through higher bid-ask spreads than environments with higher interest rates.

In the derivations leading to (24) it was assumed that market makers demand a lower risk-adjusted return than the bond market in general. But with this specific cost function, low interest rates imply higher bid-ask spreads even when market makers demand the same risk-adjusted return on their bond holdings as the bond market in general. In this case, the second term of (24) would vanish, but the first term is sufficient for the result. The mechanism is that a low interest rate in period t, all else equal, implies a higher period t bond price than when the interest rates is higher. This leads to a higher value of the market maker's period t inventory position for given N_t, and thereby increasing costs, which the market maker must be compensated for through a higher bid-ask spread.

4.3. Asymmetry at the lower bound

In this section we analyse in greater detail the impact on market liquidity of the asymmetric sample space for future interest rates, and thus future bond prices, due to the existence of a lower bound for the short rate. In the versions of the market-maker model we have considered so far, the sample space for future bond prices does not affect the equilibrium bid-ask spread directly, only through the expected risk-adjusted bond-price change. However, when market makers are loss averse, i.e. capital losses on bond holdings are extra costly, the sample space does matter for market liquidity, which we will see below.

At the lower bound, the distribution of future interest rates is asymmetric in the sense that interest rates cannot fall, only increase. Through the bond-pricing model from above, this implies that bond prices are more likely to fall than to rise, and hence that changes in bond prices are more likely to be negative than positive.

To illustrate this point we have simulated the bond-pricing model from section (4.2). For now we simulate two versions of the model. In the first, we calibrate the parameters in the model such that the average yield curve in the model resembles the average empirical yield curve prior to the outbreak of the financial crisis. This means that we calibrate the model such that the unconditional expectation, or long-run expectation, of the short rate is equal to 4.5 percent per year. As discussed above, the market price of risk, λ_0, is set such that the average yield curve is upward sloping. In the second version of model, which we from now on will call the lower-bound model, the long-run mean is assumed to be 1 percent, the initial short rate is zero, $r_0 = 0$. For further details, see appendix (A.2).[5]

We simulate these two versions of the model and calculate the future interest rate, r_{t+1}, and the future bond price, P^{n-1}_{t+1}, associated with each simulation outcome. The results are shown as cumulative distribution functions in figures (2) and (3). We see that the distribution is asymmetric at the lower bound in the sense that bond prices mostly fall. Outside the lower bound, the distribution of future bond prices is not asymmetric to the same extent, since bond price gains and bond price losses are almost equally likely.

As described above, the asymmetry in future bond-price changes has no direct effect on market liquidity in the versions of the market-maker model we have considered so far – only indirectly through the impact on expected risk-adjusted bond-price changes. However, if we expect market makers to be loss averse along the lines of Kahneman and Tversky (1979), the asymmetry will have direct effects on market liquidity through the bid-ask spread, since capital losses on their bond holding inventories will be extra costly for the market makers. Specifically, we assume that the market makers face a cost function which penalises inventory losses relative to gains. This could, among other things, reflect that capital is eroded and that the market maker's solvency can come under pressure if the market maker experiences losses on its inventory position. Hence, the cost function reflecting loss aversion is

$$E_t\left[C_{t+1}\right] = \int_{-\infty}^{\infty} max\left(0, -N_t \Delta P^{n-1}_{t+1}\right) f_{\Delta P^{n-1}_{t+1}}(x)dx, \tag{25}$$

where we have integrated over all possible states of the bond-price change. $f_{\Delta P^{n-1}_{t+1}}$ is the density function for bond-price changes. Hence, if $N_t > 0$ and the change in the bond price

[5] We also stress that although it should not, the short rate can be negative in the simulations. This happens because we use a discrete-time setup. Non-negativity of the short rate process only holds in the continuous-time limit.

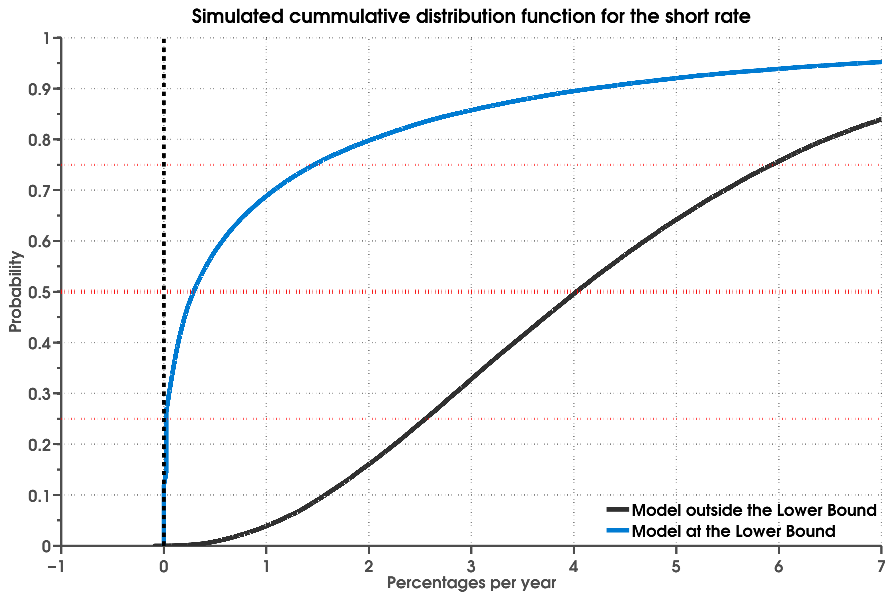

Figure 2: **Simulated cumulative distribution function for the short rate,** r_t

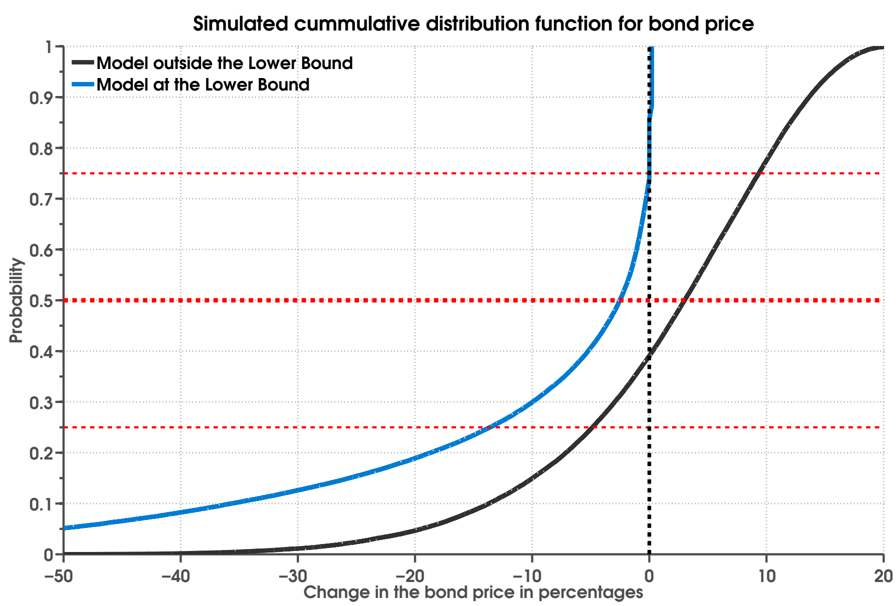

Figure 3: **Simulated cumulative distribution function for bond-price changes**

is negative, $\Delta P_{t+1}^{n-1} < 0$, meaning that the holder of the bond faces a capital loss, costs are higher. On the contrary, if the change in the bond price is positive, $\Delta P_{t+1}^{n-1} > 0$, meaning that the holder of the bond faces a capital gain, the market maker gains on its portfolio and costs are zero.

We return to the case in which market makers demand exactly the same risk-adjusted return as the bond market in general, i.e. the equilibrium bid-ask spread in (15) reduces to the first term. We do so in order to highlight that loss aversion is another possible way to generate a negative relationship between the bid-ask spread and the level of the short rate, i.e. it is not a necessary assumption that market makers demand a lower risk-adjusted return than the bond market in general. Now, we can write the equilibrium bid-ask spread as[6]

$$S_t^\star = \frac{\int_{-\infty}^{\infty} max\left(0, -N_t \Delta P_{t+1}^{n-1}\right) f_{\Delta P_{t+1}^{n-1}}(x)dx}{2V_0 P_t^n}. \tag{26}$$

To get a sense of the impact of the interest rate level on this bid-ask spread, we carry out simulations of the CIR-model similar to those depicted in figures (2) and (3). The result of the simulations are shown in figure (4).

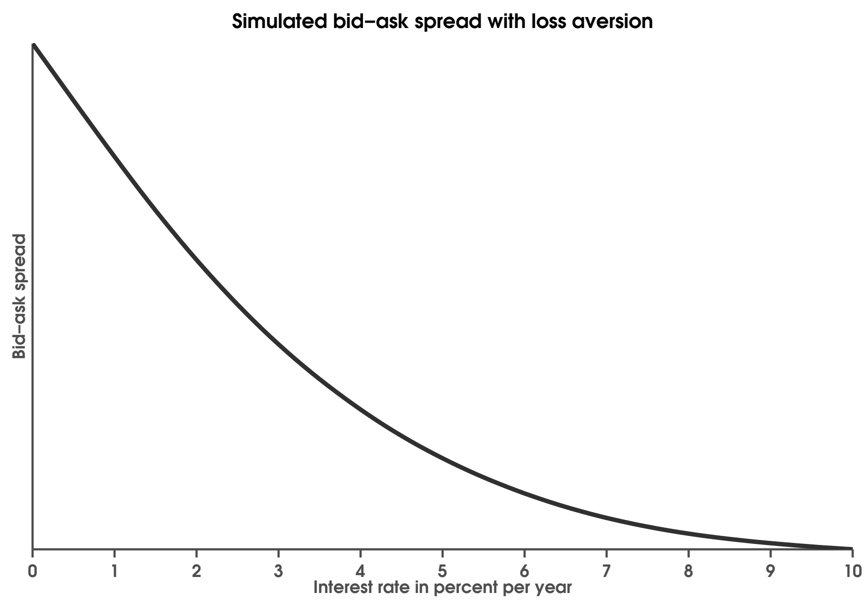

Figure 4: **Simulated bid-ask spreads with loss averse market makers**
In the figure is shown simulated equilibrium bid-ask spreads, $S_t^\star$. The bond prices are simulated using the CIR-model in which the initial short term interest rate is varied from 0 percent per year to 10 percent per year. The long-run mean of the short rate, θ, or unconditional mean is set to $4\frac{1}{2}$ percent per year in all simulations. The expected trading volume is normalised to $\frac{1}{2}$, while the initial net position is normalised to 1.

The simulations confirm the intuition from above. When the level of the short rate is far

[6] For simplicity, we introduce loss aversion in an ad-hoc fashion through the cost function. Full consistency with the bond-pricing part of our model would require that the bonds are priced under loss aversion – and not only under risk aversion. We project that this extension of the model would give rise to the same qualitative results as long as market makers are less loss averse than the bond market in general. We leave this point for future research.

away from the lower bound, bond prices are almost as likely to increase as to fall. However, when the level of the short-term interest rate is relatively closer to the lower bound, bond prices are more likely to fall. Under loss aversion this means that costs rise and hence that the bid-ask spread is higher at the lower bound. Consequently, loss aversion gives rise to a worsening of market liquidity at the lower bound for the short-term interest rate.

We lastly discuss our results when the bond in the model is a coupon bond or a portfolio of bonds. For simplicity, we have so far analysed the case of a zero-coupon bond. In this case, the duration, or the sensitivity of the bond price with respect to its yield, is always equal to the time-to-maturity, n. It is thus not possible to address the fact that duration increases when the interest rate falls.

In contrast, duration increases when the short rate falls, in the case of a coupon bond or a portfolio of bonds. This effect enhances the negative effect of low interest rates on market liquidity, all else being equal. Intuitively, when interest rates fall, the duration of a portfolio of bonds increases, as shown in appendix (A.2), figure (7) for the case of a 10 year bullet bond which pays yearly coupons. Higher duration at the lower bound makes a portfolio of bonds more risky in the sense that bond prices are more sensitive to changes in interest rates. This enhances the results, all else being equal, in the sense that if the underlying asset in the model would have been a portfolio of bonds or a coupon-bond, then when short term interest rates are low, duration would be higher and bonds more riskier.

5. Concluding remarks

In this paper, we have analysed market liquidity within a theoretical framework. Specifically, we analysed the determination of the bid-ask spread in a simple model for market making with inventory costs, which we subsequently expanded to the case of fixed-income instruments. While we are not the first to do so, the model is to our knowledge the first which has an explicit channel for capital losses and gains and thereby risk in the determination of the bid-ask spread. Furthermore, we have introduced an explicit equilibrium model for the pricing of bonds into the market maker model. This allow us to study in detail the determination of bid-ask spreads.

In the extended model we found that the bid-ask spread is adjusted to cover two terms: the expected costs at expected trading volumes and the expected (net) holding period return from holding the inventory for one period and selling it again the following period. The second term implies that expectations of a higher future bond price depress the bid-ask spread all else being equal. Specifically, higher expected compensation for holding the inventory puts downward pressure on the bid-ask spread, the intuition being that in expectation terms the market maker will gain on his market making, which through free entry and perfect competition forces the spread down such that the market maker does not lose clients. We then showed that under the assumption of no-arbitrage and in the case in which the market makers demand exactly the same risk-adjusted return as the bond market in general, the second term does not affect the bid-ask spread. The intuition is that market makers are already fully compensated for the inherent risk of changes in bond prices in risk-adjusted expectation terms through the market price of the bond, and therefore they do not pass expected capital gains or losses on to the bid-ask spread in equilibrium.

We next assumed the market makers demand less compensation for bearing risk, and assumed that bond prices are priced in a CIR-type model. We showed that in this case, low interest rate environments are characterised by higher bid-ask spreads than environments with higher interest rates. This works through the effect that the market maker extracts a positive compensation for holding risky fixed income securities, which in equilibrium results in lower bid-ask spreads, and this risk premium is increasing in the level of the interest rate. The latter observation follows from the CIR-model in which compensation for risk falls when the level of the interest rates falls, as volatility is increasing in the level of the interest rate, which there is evidence for in data.

We finally analysed market liquidity at the lower bound. We showed that at the lower bound, the distribution of bond prices is asymmetric in the sense that bond prices are most likely to fall in contrast to the case outside the lower bound where bond-price gains and bond-price losses are equally likely. We then showed that if market makers are loss averse in the sense that capital losses from holding an inventory of bonds are more costly than gains of equivalent size, bid-ask spreads are higher when the level of interest rates are lower and largest at the lower bound. This result holds even when market makers demand exactly the same risk-adjusted return as the market in general.

Everything said above concerned zero-coupon bonds. If we instead had assumed coupon-bonds, then duration would have risen when the level of interest rate was lower leading to more sensitive bond prices with respect to the short rate. We showed that duration all else equal enhances the results stated above.

We lastly point to scope for further research. The model is a partial-equilibrium model. In a general-equilibrium setup it can be expected that agents react to the larger bid-ask spread by decreasing their demand for market-making services. This will, all else equal, reduce the trading volume and thereby the size of the inventory for market makers. This will in turn imply that market makers will be less willing and able to absorb flows of assets. That can have dire consequences for market liquidity in an environment with low interest rates, where it can be expected that many investors would want to sell bonds in case of news about future monetary policy. As an example, market makers reduced considerably their inventories in response to the so-called taper tantrum and widened their bid-ask spreads. Lastly, in this paper we have only looked at market liquidity and not the interdependence between funding liquidity for market liquidity – a well-functioning funding market is a prerequisite for the market making in the model. We leave this for future research.

References

Amihud, Y. (2002). Illiquidity and stock returns: cross-section and time-series effects. *Journal of financial markets 5*(1), 31–56.

Backus, D., S. Foresi, and C. Telmer (1998). Discrete-time models of bond pricing. Technical report, National Bureau of Economic Research.

Brunnermeier, M. K. and L. H. Pedersen (2009). Market liquidity and funding liquidity. *Review of Financial studies 22*(6), 2201–2238.

Buchholst, B. V., J. Gyntelberg, and T. Sangill (2010). Liquidity of danish government and covered bonds – before, during and after the financial crisis – preliminary findings. *Danmarks Nationalbank Working Papers 70*.

Campbell, J. Y., A. W. Lo, and A. C. McKinley (1997). *The Econometrics of Financial Markets*. Princeton University Press.

Committee on the Global Financial System (2014). Market-making and proprietary trading: Industry trends, drivers and policy implications. *CGFS Papers 52*.

Cox, J. C., J. E. Ingersoll, and S. A. Ross (1985). A theory of the term structure of interest rates. *Econometrica 53*(2), 385–407.

Danmarks Nationalbank (2015). *Finansiel stabilitet, 2. halvår*. Danmarks Nationalbank, in Danish but forthcoming in English: Financial stability, 2nd half.

Dick-Nielsen, J., J. Gyntelberg, and J. Lund (2014). From funding liquidity to market liquidity: Evidence from danish bond markets. In *Midwest Finance Association 2014 Annual Meeting*.

Fender, I. and U. Lewrick (2015). Shifting tides-market liquidity and market-making in fixed income instruments. *BIS Quarterly Review March*.

Foucault, T., M. Pagano, and A. Röell (2013). *Market Liquidity: Theory, Evidence, and Policy*. Oxford University Press.

IMF (2015). Global financial stability report, ch 2. market liquidity resilient or fleeting?

Kahneman, D. and A. Tversky (1979). Prospect theory: An analysis of decision under risk. *Econometrica: Journal of the Econometric Society*, 263–291.

Parameswaran, S. K. (2003). *Futures Markets: Theory and Practice*, Volume 1. Tata MacGraw-Hill.

Pedersen, J. (2015). The danish natural real rate of interest and secular stagnation. *Danmarks Nationalbank Working Papers 94*.

Piazzesi, M. (2003). Affine term structure models. *Handbook of Financial Econometrics 1*, 691–766.

Roll, R. (1984). A simple implicit measure of the effective bid-ask spread in an efficient market. *The Journal of Finance 39*(4), 1127–1139.

Shen, P. and R. M. Starr (2002). Market-makers' supply and pricing of financial market liquidity. *Economics letters 76*(1), 53–58.

Stoll, H. R. (1978). The supply of dealer services in securities markets. *The Journal of Finance 33*(4), 1133–1151.

Sun, T. (1992). Real and nominal interest rates: a discrete-time model and its continuous-time limit. *Review of Financial Studies 5*(4), 581–611.

A. The model for the price of a zero-coupon bond

In what follows, it is assumed that the short interest rate follows a Cox-Ingersoll-Ross process within the affine term structure theory, see Cox et al. (1985) and Backus et al. (1998),

$$r_{t+1} = \left(1-\phi\right)\theta + \phi r_t + \sigma\sqrt{r_t}\varepsilon_{t+1}, \tag{27}$$

with $\varepsilon_{t+1} \sim N(0,1)$. The only source of uncertainty in the model is the short rate; the stochastic term ε_{t+1}. Time is in months, so the short rate is the monthly rate. We will in what follows need to calculate conditional and unconditional first and second moments, which are given by the following expressions:

$$\begin{aligned} E_t\left[r_{t+1}\right] &= \left(1-\phi\right)\theta + \phi r_t \\ E\left[r\right] &= \theta \\ V_t\left[r_{t+1}\right] &= \sigma^2 r_t \\ V\left[r\right] &= \theta\frac{\sigma^2}{1-\phi^2}. \end{aligned} \tag{28}$$

From these expressions θ can be interpreted as the long-run mean of the short rate. Volatility, or the conditional second moment, is time-varying as it depends on the short rate meaning that uncertainty is increasing in the level of the short rate, which also can be found in data, see Piazzesi (2003). Finally, ϕ determines the persistence in the short rate.

We have chosen this specification as it rules out negative interest rates. Recent events have shown that interest rate can be negative. Here we simply want a specification for the short rate with a floor; it is not important whether that floor is zero or slightly below.

The bond price, P_t^n, with time-to-maturity n is priced as follows

$$\begin{aligned} P_t^n &= e^{-A_n - B_n r_t} \\ A_{n+1} &= A_n + B_n\left(1-\phi\right)\theta, A_0 = A_1 = 0 \\ B_{n+1} &= \left(1+\lambda_0^2/2 + B_n\phi\right) - \left(\lambda_0 + B_n\sigma\right)^2/2, B_0 = 0, B_1 = 1, \end{aligned}$$

and interest yields are given by $y_t^n = -\frac{1}{n}log(P_t^n)$. A_n determines the shape of the average yield curve. B_n determines by how much the yield curve changes if the short rate changes; it is the elasticity of the yield curve with respect to the short rate, which is also the duration: $\frac{\partial log\left(P_t^n\right)}{\partial r_t} = -B_n$.

The recursions can be found through the standard asset-pricing formula under complete and arbitrage-free financial markets that today's asset price is equal to the expected pay-off discounted by the stochastic discount factor, M_{t+1}:

$$\begin{aligned} P_t^n &= E_t\left[M_{t+1}P_{t+1}^{n-1}\right] \\ exp(-A_n - B_n r_t) &= E_t\left[exp(-r_t)exp(-\frac{1}{2}\sigma^2\Lambda^2 - \Lambda\sigma\varepsilon_{t+1}))exp(-A_{n-1} - B_{n-1}r_{t+1})\right], \end{aligned}$$

where we have included the stochastic discount factor, $M_{t+1} = exp(-r_t)exp(-\frac{1}{2}\sigma^2\Lambda^2 - \Lambda\sigma\varepsilon_{t+1}))$ in the second line. By matching coefficients and using the fact that one krone today is equal to one krone, the recursions A_{n+1} and B_n can be found.

In the model for the market maker, we will need to analyse the expected holding period return from holding a bond for one-period, $E_t\left[\frac{P_{t+1}^{n-1}}{P_t^n}\right]$. The expression for this term within the CIR-model is found below.

Result 1:

$$E_t\left[\frac{P_{t+1}^{n-1}}{P_t^n}\right] = exp\left(r_t\right) exp\left(-\lambda_0 B_{n-1}\sigma r_t\right) \tag{29}$$

Proof.

Substituting the theoretical bond prices into this definition, we get:

$$\begin{aligned} E_t\left[\frac{P_{t+1}^{n-1}}{P_t^n}\right] &= E_t\left[exp\left(-A_{n-1} - B_{n-1}r_{t+1}\right) exp\left(A_n + B_n r_t\right)\right] \\ &= exp\left(-A_{n-1} + A_n + B_n r_t - B_{n-1}\left[\left(1-\phi\right)\theta + \phi r_t\right] + \frac{\left(B_{n-1}\sigma\right)^2}{2} r_t\right) \\ &= exp\left(\left(B_n - B_{n-1}\phi + \frac{\left(B_{n-1}\sigma\right)^2}{2}\right) r_t\right) \\ &= exp\left(\left(1 + \lambda_0^2/2 - \left(\lambda_0 + B_{n-1}\sigma\right)^2/2 + \frac{\left(B_{n-1}\sigma\right)^2}{2}\right) r_t\right) \\ &= exp\left(r_t\right) exp\left(-\lambda_0 B_{n-1}\sigma r_t\right), \end{aligned}$$

where expression (28) has been used in the second line. The third line has made use of $A_n = A_{n-1} + B_{n-1}\left(1-\phi\right)\theta$, and in the forth line we have used that $B_n - B_{n-1}\phi = 1 + \lambda_0^2/2 - \left(\lambda_0 + B_n\sigma\right)^2/2$.

■

A.1. Two models for the short interest rate

We will evaluate the bond-pricing model through simulations. We make two calibrations. In the calibration outside the lower bound we use the following strategy. θ is as discussed above the unconditional expectation, or long-run expectation, of the short rate; $\theta = \frac{4.5}{1200}$. Short rates are very persistent bordering unit-root behaviour; ϕ=0.975. Finally, we set the volatility such that the unconditional variance of the short rate is 3 percent per year;

$\theta \frac{\sigma^2}{1-\phi^2} = \left(\frac{3}{1200}\right)^2$. The market price of risk, λ_0 is set such that the average yield curve is upward sloping as normally is the case in data; that is $\lambda_0 = -1$.

We also want to analyse the behaviour of bid-ask spread at the lower bound. To do so, it would perhaps have been more appropriate to use a multi-factor model that could have incorporated shocks to the long-run value of the short rate, $\theta \rightarrow \theta_t = \rho\theta_{t-1} + \sigma^{\theta}\varepsilon_t^{\theta}$. That could as an example capture the fall of the long-run natural real rate, see as an example Pedersen (2015).

However, we have chosen to capture this economic environment in a different calibration instead which we from now on will denote the ZLB-model. We stress that the lower bound can be, and surely is, different from 0. But that is not crucial for any of the results in this paper. In the ZLB-model, the parameters are calibrated as follows. The long-run mean is assumed to be 1 pct.; $\theta = \frac{1}{1200}$, the actual short rate is zero, r_0, while persistence is assumed to be unchanged with respect to the model in "normal" times, $\phi = 0.975$. We set volatility and price of risk equal to same values as in the model for normal times. [7]

A.2. Properties of the CIR model inside and outside the lower bound

We will in this section present some properties of the CIR-model for the yield curve both inside and outside the ZLB. In figure (5) is shown simulated paths for the short term interest rate in the two models.

The short term interest rates fluctuate around the unconditional mean, θ. The size of the fluctuations is determined by the volatility parameter, σ. In the top figure, the unconditional mean is calibrated to $3\frac{1}{2}$ percent per year while in the bottom figure it is around 1 percent reflecting that at the ZLB, the natural, or long-run, interest rate has fallen as well. But the short term rate does not hit zero, which in the market maker model is assumed to be the lower bound for short term policy rates.

In figure (6(a)) are shown examples of yield curves produced by the CIR-model. As explained in the previous section, yields of bonds with different maturities are priced through no-arbitrage - it is possible to get higher expected bond returns only by taking on more risk; that is, the Sharpe-ratio is equal for bonds with different maturities. The coefficients A_n and B_n are the results of this idea.

As explained above, the factor loadings B_n determine by how much the bond price falls as the interest rate increases. This has implications for the expected holding period return across maturities. The factor loadings are shown in figure (6(b)). In the calibration we have increased the price of risk, Λ, in the model for the lower bound to be twice as big as in the model outside the lower bound. This is also the case for the volatility. This reflects the elevated level of risk in the economy as well as increasing risk aversion. As the recursions B_n neither depend on the unconditional short rate, θ, nor on the current short rate, these

[7] We also stress that although it should not, the short rate can be negative in the simulations. That is because we use a discrete-time setup. Non-negativity of the short rate process only holds in the continuous time limit.

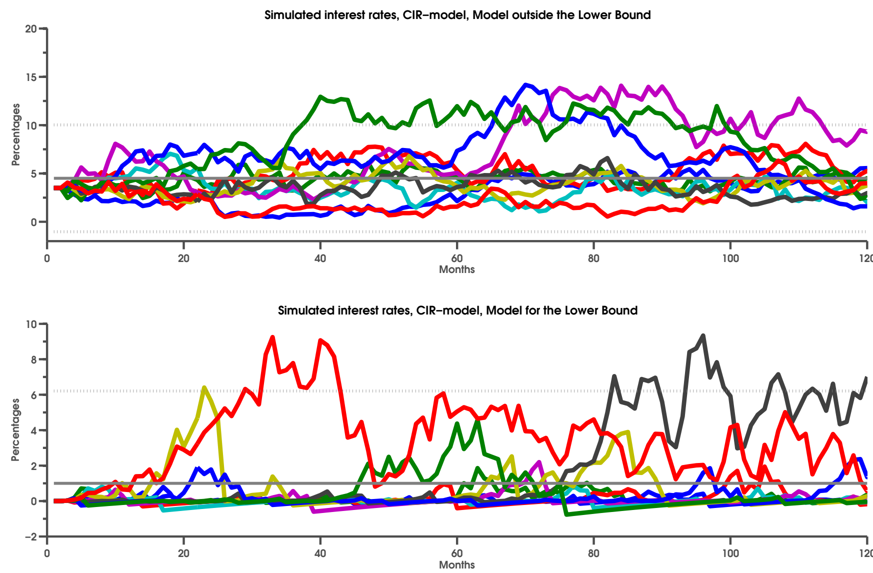

Figure 5: **Simulated interest rates**
In the figure is shown simulated path for the short term interest rate in the CIR model inside and outside the ZLB in the two versions of the model.

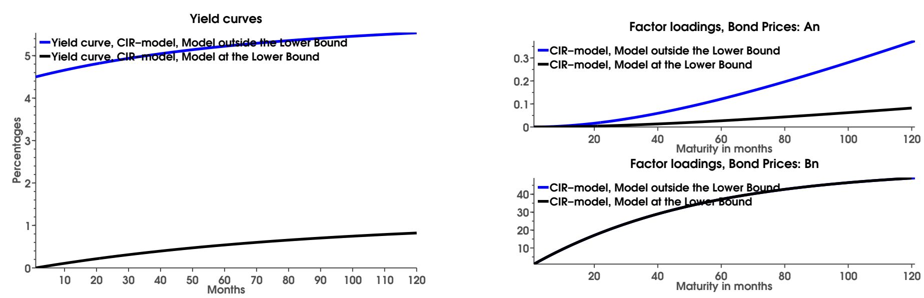

Figure 6: **Average yield curve and Factor loadings**
In the top figure is shown examples of yield curves from the CIR-model inside and outside the lower bound for the short rate. In the figure is shown the factor loadings, that is the recursions A_n and B_n derived above, for the two models: The model for the short interest rate outside and inside the lower bound.

choices explained the different shape of B_n across the two models.

Duration is constant and equal to the time-to-maturity for a zero-coupon bond. In figure (7) is shown how duration varies for a coupon bond when the short term interest rate is varied from 0 percent per year to 10 percent per year.

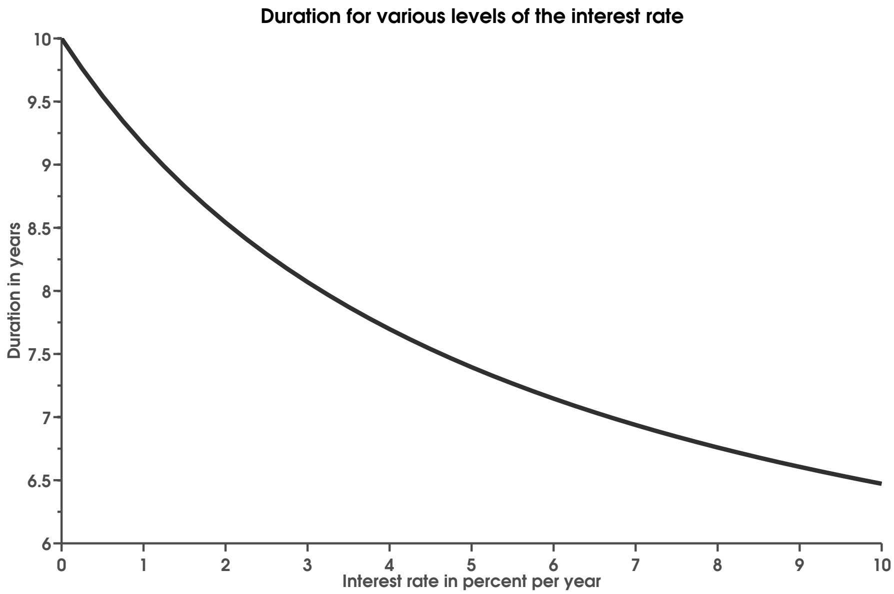

Figure 7: **Duration of a coupon bond**
In the figure is shown approximate duration for a coupon bond. The bond prices are calculated using the CIR-model in which the initial short term interest rate is varied from 0 percent per year to 10 percent per year. Coupons are paid yearly until year 10 in which the principal of 100 is paid.

B. Derivations used in the expressions for the bid-ask spread under assumptions about costs

B.1. Quadratic cost function

In this subsection we assume a quadratic cost function

$$C_{t+1} \equiv \eta \left(P_{t+1}^{n-1} N_{t+1} \right)^2, \tag{30}$$

where $\eta > 0$. Since the shock to the short rate, ε, follows a normal distribution, the bond price follows a log-normal distribution. Therefore, the expected value and variance of the next period bond price can be written

$$E_t \left[P_{t+1}^{n-1} \right] = e^{-A_{n-1} - B_{n-1} \left[(1-\phi)\theta + \phi r_t \right] + \frac{1}{2} (B_{n-1\sigma})^2 r_t} \tag{31}$$

$$V_t \left[P_{t+1}^{n-1} \right] = \left(e^{(B_{n-1}\sigma)^2 r_t} - 1 \right) e^{-2\left(A_{n-1} + B_{n-1} \left[(1-\phi)\theta + \phi r_t \right] \right) + (B_{n-1}\sigma)^2 r_t}. \tag{32}$$